

Time-dependent Decoherence-Free Subspace

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With time-dependent Lindblad operators, an open system may have a time-dependent decoherence-free subspace (t-DFS). In this paper, we define the t-DFS and present a necessary and sufficient condition for the t-DFS. Two examples are presented to illustrate the t-DFS, which show that this t-DFS is not trivial, when the dimension of the t-DFS varies.

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I. INTRODUCTION

In recent years, many proposals are presented to protect quantum systems against decoherence. Except the method to weaken the coupling between the system and its surroundings, these proposals include the dynamical decoupling [1–3], quantum error-correcting codes [4, 5], the scheme based on the decoherence-free subspaces (DFSs) or noiseless subsystems [6–9], and the scheme based on the quantum reservoir engineering [10, 11].

The DFS is defined as a subspace within which the system undergoes an unitary evolution [9]. It was experimentally realized in a variety of systems [12–14] and has drawn much attention because its potential applications in quantum information processing [15, 16]. Generally speaking, there are two ways to define the DFS. The first is given in [7], where the DFS includes all states that each state ρ satisfies $\mathcal{L}(\rho) = 0$, where $\mathcal{L}(\dots)$ represents the Lindblad superoperator (we call it as the first definition of DFS). The second definition was given in [9], which is formulated as follows. Let the time evolution of an open system with Hilbert space \mathcal{H}_S be governed by the Markovian master equation. A decoherence-free subspace \mathcal{H}_{DFS} is defined as a subspace of \mathcal{H}_S such that all states $\rho(t)$ in DFS fulfill $\partial_t \text{Tr}[\rho^2(t)] = 0$, for $\forall t \geq 0$, with $\text{Tr}[\rho^2(0)] = 1$. By this definition, it was proved that the subspace $\mathcal{H}_{DFS} = \text{Span}\{|\Phi_1\rangle, |\Phi_2\rangle, \dots, |\Phi_M\rangle\}$ is a DFS if and only if each basis of \mathcal{H}_{DFS} satisfies $F_\alpha|\Phi_j\rangle = c_\alpha|\Phi_j\rangle, j = 1, \dots, M; \alpha = 1, \dots, K$, and \mathcal{H}_{DFS} is invariant under $H_{\text{eff}} = H + \frac{i}{2} \sum_\alpha (c_\alpha^* F_\alpha - c_\alpha F_\alpha^\dagger)$, where H is the Hamiltonian of the open quantum system.

Notice that the basis of the aforementioned DFS is time-independent, the DFS is then time-independent. For time-dependent Lindblad operators, however, a time-independent DFS may not exist, then a natural question arises: What is the DFS for open systems with time-dependent Lindblad operators? If the DFS is time-dependent, what is the condition for the system to remain in this subspace?

In this paper, we will give a detail analysis for a time-dependent DFS (t-DFS). The analysis is given based on

the Lindblad master equation with time-dependent Lindblad operators. We shall adopt the second definition for the t-DFS and develop a theorem to give a necessary and sufficient condition for the t-DFS.

This paper is organized as follows. In Sec.II, we shall give the definition for the t-DFS and derive a necessary and sufficient condition for it. In Sec.III, we present two examples to illustrate the t-DFS, showing that the necessary and sufficient condition can be satisfied by manipulating the Hamiltonian. Finally we conclude by summarizing our results in Sec.IV.

II. NECESSARY AND SUFFICIENT CONDITION FOR TIME-DEPENDENT DFS

In this section, we will present a necessary and sufficient condition for the t-DFS and show that this condition can be satisfied by manipulating the Hamiltonian of the open system.

Consider a system S with N -dimensional Hilbert space \mathcal{H}_S coupling with an environment. The time evolution of the open system is assumed to be governed by,

$$\begin{aligned} \dot{\rho}(t) &= -i[H(t), \rho(t)] + \mathcal{L}(t)\rho(t), \\ \mathcal{L}(t)\rho(t) &= \sum_\alpha \left[F_\alpha(t)\rho(t)F_\alpha^\dagger(t) - \frac{1}{2}\{F_\alpha^\dagger(t)F_\alpha(t), \rho(t)\} \right], \end{aligned} \quad (1)$$

where $F_\alpha(t)$ ($\alpha = 1, 2, 3, \dots, K$) are time-dependent Lindblad operators and $\mathcal{L}(t)$ describes the Lindblad superoperator of the open quantum system.

Theorem 1. *Let the time evolution of an open quantum system in a finite-dimensional Hilbert space be governed by Eq.(1) with time-dependent Hamiltonian $H(t)$ and time-dependent Lindblad operators $F_\alpha(t)$. The subspace*

$$\mathcal{H}_{DFS}(t) = \text{Span}\{|\Phi_1(t)\rangle, |\Phi_2(t)\rangle, \dots, |\Phi_M(t)\rangle\} \quad (2)$$

is a t-DFS if and only if each basis vector of $\mathcal{H}_{DFS}(t)$ satisfies

$$F_\alpha(t)|\Phi_j(t)\rangle = c_\alpha(t)|\Phi_j(t)\rangle, j = 1, \dots, M; \alpha = 1, \dots, K, \quad (3)$$

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and $\mathcal{H}_{DFS}(t)$ is invariant under

$$H_{\text{eff}}(t) = G(t) + H(t) + \frac{i}{2} \sum_{\alpha} (c_{\alpha}^*(t) F_{\alpha}(t) - c_{\alpha}(t) F_{\alpha}^{\dagger}(t)). \quad (4)$$

Here $G(t) = iU^{\dagger}(t)\dot{U}(t)$ and $U(t)$ is an unitary operator

$$U(t) = \sum_{j=1}^M |\Phi_j(0)\rangle \langle \Phi_j(t)| + \sum_{n=1}^{N-M} |\Phi_n^{\perp}(0)\rangle \langle \Phi_n^{\perp}(t)|. \quad (5)$$

Proof. Firstly, notice that the effect of $U(t)$ is to map a set of time-dependent bases of \mathcal{H}_S into a time-independent one. Transforming the density matrix $\rho(t)$ into a rotating frame, i.e., $\bar{\rho}(t) = U(t)\rho(t)U^{\dagger}(t)$, we write the master equation Eq.(1) as,

$$\frac{\partial \bar{\rho}(t)}{\partial t} = -i [\bar{H}(t), \bar{\rho}(t)] - i\bar{G}(t)\bar{\rho}(t) + i\bar{\rho}(t)\bar{G}(t) + \bar{\mathcal{L}}(t)\bar{\rho}(t), \quad (6)$$

where,

$$\bar{H}(t) = U(t)H(t)U^{\dagger}(t),$$

$$\bar{\mathcal{L}}(t)\bar{\rho}(t) = \sum_{\alpha} (\bar{F}_{\alpha}(t)\bar{\rho}(t)\bar{F}_{\alpha}^{\dagger}(t) - 1/2\{\bar{F}_{\alpha}^{\dagger}(t)\bar{F}_{\alpha}(t), \bar{\rho}(t)\})$$

with $\bar{F}_{\alpha}(t) = U(t)F_{\alpha}(t)U^{\dagger}(t)$. Clearly, $\bar{G}(t) = U(t)G(t)U^{\dagger}(t) = i\dot{U}(t)U^{\dagger}(t) = -iU(t)\dot{U}^{\dagger}(t) = \bar{G}^{\dagger}(t)$, this indicates that $\bar{G}(t)$ is a Hermitian operator. By defining new Lindblad operator as $\tilde{F}_{\alpha}(t) = \bar{F}_{\alpha}(t) - c_{\alpha}(t)$, the decoherence terms in Eq.(6) can be rewritten as

$$\begin{aligned} \bar{\mathcal{L}}(t)\bar{\rho}(t) &= \tilde{\mathcal{L}}(t)\bar{\rho}(t) \\ &- i \left[\frac{i}{2} \sum_{\alpha} (c_{\alpha}^*(t)\bar{F}_{\alpha}(t) - c_{\alpha}(t)\bar{F}_{\alpha}^{\dagger}(t)), \bar{\rho}(t) \right], \end{aligned} \quad (7)$$

where

$$\tilde{\mathcal{L}}(t)\bar{\rho}(t) = \sum_{\alpha} (\tilde{F}_{\alpha}(t)\bar{\rho}(t)\tilde{F}_{\alpha}^{\dagger}(t) - 1/2\{\tilde{F}_{\alpha}^{\dagger}(t)\tilde{F}_{\alpha}(t), \bar{\rho}(t)\}).$$

Note that the requirement of invariance of the subspace $\mathcal{H}_{DFS}(t)$ under the operator $H_{\text{eff}}(t)$ in Theorem 1 implies $\langle \Phi_n^{\perp}(0) | \bar{H}_{\text{eff}}(t) | \Phi_j(0) \rangle = \langle \Phi_n^{\perp}(t) | H_{\text{eff}}(t) | \Phi_j(t) \rangle = 0$ for $\forall n, j$. In the rotating frame, the subspace \mathcal{H}_{DFS} spanned by $\{|\Phi_j(0)\rangle\}$ must be invariant under the operator

$$\begin{aligned} \bar{H}_{\text{eff}}(t) &= \bar{H}(t) + \bar{G}(t) \\ &+ \frac{i}{2} \sum_{\alpha} (c_{\alpha}^*(t)\bar{F}_{\alpha}(t) - c_{\alpha}(t)\bar{F}_{\alpha}^{\dagger}(t)). \end{aligned} \quad (8)$$

Now we prove that the condition is sufficient. Any state $|\varphi(t)\rangle \in \mathcal{H}_{DFS}(t)$ can be expanded by $\{|\Phi_j(t)\rangle\}$,

$$|\varphi(t)\rangle = \sum_{j=1}^M a_j(t) |\Phi_j(t)\rangle. \quad (9)$$

By defining $|\bar{\varphi}(t)\rangle = U(t)|\varphi(t)\rangle$, $\bar{\rho}(t)$ and $\bar{\mathcal{L}}(t)\bar{\rho}(t)$ will be written as,

$$\begin{aligned} \bar{\rho}(t) &= U(t)|\varphi(t)\rangle \langle \varphi(t)| U^{\dagger}(t), \\ \bar{\mathcal{L}}(t)\bar{\rho}(t) &= -i \left[\frac{i}{2} \sum_{\alpha} (c_{\alpha}^*(t)\bar{F}_{\alpha}(t) - c_{\alpha}(t)\bar{F}_{\alpha}^{\dagger}(t)), \bar{\rho}(t) \right], \end{aligned} \quad (10)$$

where $\tilde{F}_{\alpha}(t)|\bar{\varphi}(t)\rangle = 0 \times |\bar{\varphi}(t)\rangle = 0$ and $\tilde{\mathcal{L}}(t)\bar{\rho}(t) = 0$ have been used. Hence the evolution of $\bar{\rho}(t)$ is governed by

$$\frac{\partial \bar{\rho}(t)}{\partial t} = -i[\bar{H}_{\text{eff}}(t), \bar{\rho}(t)], \quad (11)$$

so

$$\begin{aligned} \frac{\partial}{\partial t} \text{Tr}[\rho^2(t)] &= \frac{\partial}{\partial t} \text{Tr}[\bar{\rho}^2(t)] = 2\text{Tr}[\bar{\rho}(t)\dot{\bar{\rho}}(t)] \\ &= -2i\text{Tr}\{\bar{H}_{\text{eff}}(t), \bar{\rho}(t)\} \bar{\rho}(t) = 0. \end{aligned} \quad (12)$$

Thus the conditions Eqs.(3) and (4) are sufficient for $\frac{\partial}{\partial t} \text{Tr}[\rho^2(t)] = 0$.

In order to prove that the conditions are necessary, we assume that the set of basis $\{|\Phi_j(t)\rangle\}$ spans a t-DFS. At a fixed instant t_0 , the quantum state $|\varphi(t_0)\rangle$ embeds in t-DFS

$$\begin{aligned} |\varphi(t_0)\rangle &= \sum_{j=1}^M a_j(t_0) |\Phi_j(t_0)\rangle, \\ \rho(t_0) &= |\varphi(t_0)\rangle \langle \varphi(t_0)|, \end{aligned} \quad (13)$$

which implies

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \text{Tr}[\rho^2(t)]|_{t=t_0} = 2\text{Tr}[\rho(t_0) \mathcal{L}(t_0) \rho(t_0)] \\ &= 2\langle \varphi(t_0) | \mathcal{L}(t_0) \rho(t_0) | \varphi(t_0) \rangle. \end{aligned} \quad (14)$$

Then, by using Eq.(1), we obtain

$$\begin{aligned} 0 &= \langle \varphi(t_0) | \mathcal{L}(t_0) \rho(t_0) | \varphi(t_0) \rangle \\ &= \sum_{\alpha} \gamma_{\alpha} [\langle \varphi(t_0) | F_{\alpha}(t) | \varphi(t_0) \rangle \langle \varphi(t_0) | F_{\alpha}^{\dagger}(t) | \varphi(t_0) \rangle \\ &\quad - \langle \varphi(t_0) | F_{\alpha}^{\dagger}(t) F_{\alpha}(t) | \varphi(t_0) \rangle]. \end{aligned} \quad (15)$$

Without loss of generality, we set $F_{\alpha}(t_0)|\varphi(t_0)\rangle = c_{\alpha}(t_0)|\varphi(t_0)\rangle + |\varphi^{\perp}(t_0)\rangle$ with $|\varphi^{\perp}(t_0)\rangle$ being a (non-normalized) state orthogonal to state $|\varphi(t_0)\rangle$. Submitting this into Eq.(15), we have

$$\sum_{\alpha} \gamma_{\alpha} \langle \varphi^{\perp}(t_0) | \varphi^{\perp}(t_0) \rangle = 0. \quad (16)$$

Since $\gamma_{\alpha} > 0$ for $\forall \alpha$, we have $\langle \varphi^{\perp}(t_0) | \varphi^{\perp}(t_0) \rangle = 0$. It straightforwardly follows $F_{\alpha}(t_0)|\varphi(t_0)\rangle = c_{\alpha}(t_0)|\varphi(t_0)\rangle$.

Next, we prove by contradiction that the basis vectors $\{|\Phi_j(t_0)\rangle\}$ are the eigenstates of the Lindblad operators $F_{\alpha}(t_0)$ with the same eigenvalues $c_{\alpha}(t_0)$. Suppose that two arbitrary eigenvectors of the Lindblad operator $|\Phi_k(t)\rangle$ and $|\Phi_{k'}(t)\rangle$ have different eigenvalues, i.e.,

$$\begin{aligned} F_{\alpha}(t) |\Phi_k(t)\rangle &= c_{\alpha,k}(t) |\Phi_k(t)\rangle, \\ F_{\alpha}(t) |\Phi_{k'}(t)\rangle &= c_{\alpha,k'}(t) |\Phi_{k'}(t)\rangle. \end{aligned} \quad (17)$$

The state $|\phi(t)\rangle = (|\Phi_k(t)\rangle + |\Phi_{k'}(t)\rangle)/\sqrt{2}$ must be not an eigenstate of $F_\alpha(t)$. However, since $|\phi(t)\rangle$ is a state lying in $\mathcal{H}_{\text{DFS}}(t)$, the state should fulfill $\langle \mathcal{L}[|\phi(t)\rangle\langle\phi(t)|] \rangle = 0$. Hence the eigenvalues must be equal for all basis vectors.

On the other hand, according to Eq.(7), for $\bar{\rho}(t) = U(t)|\varphi(t)\rangle\langle\varphi(t)|U^\dagger(t)$ with $\bar{F}_\alpha(t)|\bar{\varphi}(t)\rangle = c_\alpha(t)|\bar{\varphi}(t)\rangle$, we have

$$\frac{\partial \bar{\rho}(t)}{\partial t} = -i\bar{G}(t)\bar{\rho}(t) + i\bar{\rho}(t)\bar{G}(t) - i[\bar{H}(t), \bar{\rho}(t)]$$

and

$$\begin{aligned} & + \bar{\mathcal{L}}(t)\bar{\rho}(t) \\ & = -i[\bar{H}_{\text{eff}}(t), \bar{\rho}(t)]. \end{aligned} \quad (18)$$

$$\bar{\rho}(t) = \mathcal{T} \exp \left[-i \int_0^t \bar{H}_{\text{eff}}(\tau) d\tau \right] \bar{\rho}(0) \mathcal{T} \exp \left[i \int_0^t \bar{H}_{\text{eff}}(\tau) d\tau \right], \quad (19)$$

where \mathcal{T} is the time-order operator. If $|\bar{\psi}(t)\rangle = \bar{H}_{\text{eff}}(t)|\bar{\varphi}(t)\rangle$ is still in subspace \mathcal{H}_{DFS} , $|\bar{\varphi}(t)\rangle$ will always be in \mathcal{H}_{DFS} . Thus, $|\varphi(t)\rangle$ will evolve unitarily in $\mathcal{H}_{\text{DFS}}(t)$, if $H_{\text{eff}}(t)|\varphi(t)\rangle$ is still a superposition of the basis vectors $\{|\Phi_j(t)\rangle\}$ of the t-DFS. Hence the conditions are necessary for $\frac{\partial}{\partial t} \text{Tr}[\rho^2(t)] = 0$. \square

Remark. Examining the unitary transformation Eq.(5), one may suspect that the t-DFS is the same as (time-independent) DFS, in the sense that a state in the t-DFS $\mathcal{H}_{\text{DFS}}(t)$ can be obtained from the time-independent DFS \mathcal{H}_{DFS} by the unitary transformation $U(t)$. This is not true, because for a t-DFS, its dimension can change with time. When the dimension of the t-DFS changes, we can not transfer the time-independent DFS into a t-DFS by an unitary transformation. In Sec.III, we will present an example to illustrate this *remark*.

We now show how to realize a t-DFS by the Theorem 1. Suppose that there is a set of degenerated eigenstates $\{|\Phi_j(t)\rangle\}$ of all Lindblad operators $F_\alpha(t)$. As stated above, if $\mathcal{H}_{\text{DFS}}(t)$ is a t-DFS, it must be invariant under $H_{\text{eff}}(t)$, i.e.,

$$\langle \Phi_n^\perp(t) | H_{\text{eff}}(t) | \Phi_j(t) \rangle = 0, \forall n, j. \quad (20)$$

Substituting Eq.(4) into Eq.(20) and considering $F_\alpha(t)|\Phi_j(t)\rangle = c_\alpha(t)|\Phi_j(t)\rangle$, we obtain

$$\begin{aligned} & -\langle \Phi_n^\perp(t) | \dot{\Phi}_j(t) \rangle - i\langle \Phi_n^\perp(t) | H(t) | \Phi_j(t) \rangle \\ & - \frac{1}{2} \sum_\alpha \gamma_\alpha c_\alpha(t) \langle \Phi_n^\perp(t) | F_\alpha^\dagger(t) | \Phi_j(t) \rangle = 0. \end{aligned} \quad (21)$$

Once the evolution of the Lindblad operators $F_\alpha(t)$ are fixed, the basis vectors of $\mathcal{H}_{\text{DFS}}(t)$ are specified. The task to achieve an unitary evolution in the t-DFS relies entirely on the Hamiltonian of the open system,

$$\begin{aligned} & \langle \Phi_k(t) | H(t) | \Phi_n^\perp(t) \rangle = -i\langle \dot{\Phi}_k(t) | \Phi_n^\perp(t) \rangle \\ & - \frac{i}{2} \sum_\alpha \gamma_\alpha c_\alpha^*(t) \langle \Phi_k(t) | F_\alpha(t) | \Phi_n^\perp(t) \rangle. \end{aligned} \quad (22)$$

In the next section, we will present two examples to show that the quantum state of the open system can be restricted to evolve unitarily in the t-DFS by manipulating the Hamiltonian according to Eq.(22).

III. EXAMPLES

In this section, we will present two examples. In the first example, we show that the sufficient and necessary condition for the t-DFS provides us a way to keep the open system in the t-DFS by manipulating the system Hamiltonian. In contrast to the scheme in Ref.[17], where the adiabatic condition is required to maintain the system with high fidelity in the t-DFS, here the adiabatic condition is removed. Here we should mention that some effort has been done on such issue, in which the time-dependent Lindblad operators evolves unitarily[18]. However, for the t-DFS, such a limit on the evolution of Lindblad operators is not necessary, therefore t-DFS we proposed is general. From the other aspect, by the first example, we explain why the adiabatic condition is required in Ref.[17]—the Hamiltonian in that paper does not satisfy the condition of t-DFS. It seems that the t-DFS can be found by an unitary transformation, this is not the case when the dimension of the t-DFS changes, this point will be shown in the second example.

A. Driven Ξ -type three-level atom coupled to a broadband time-dependent squeezing vacuum reservoir

Consider a Ξ -type atom coupled to a time-dependent broadband squeezed vacuum field [19, 20]. For squeezed vacuum field, the initial state can be written as

$$|vac(\eta)\rangle = K(\eta)|vac\rangle,$$

where $K(\eta)$ is a multi-mode squeezing transformation. There are three classical fields $\Omega_j(t)$ ($j = 1, 2, 3$) interact-

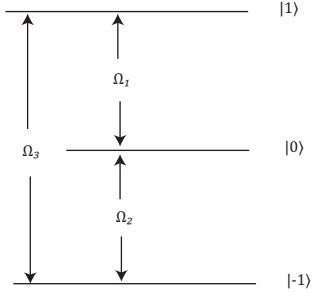


FIG. 1: (Color online) Schematic energy levels. A Ξ -type three-level system driven by three classical fields $\Omega_j(t)$. We treat this system as an open system since we will consider it coupled to a broadband squeezed vacuum reservoir.

ing with the atom as shown in FIG.1. Under the Born-Markovian approximation, the evolution of the atom is governed by the following master equation [19],

$$\dot{\rho}(t) = -i[H(t), \rho(t)] - \frac{\gamma}{2}\{R^\dagger(\eta)R(\eta)\rho(t) + \rho(t)R^\dagger(\eta)R(\eta) - 2R(\eta)\rho(t)R^\dagger(\eta)\}, \quad (23)$$

where $H(t) = [\Omega_1(t)|1\rangle\langle 0| + \Omega_2(t)|0\rangle\langle -1| + \Omega_3(t)|1\rangle\langle -1| + h.c.]$, and

$$R(\eta) = S \cosh(r) + \exp(i\phi)S^\dagger \sinh(r). \quad (24)$$

The operator $S = |1\rangle\langle 0| + |0\rangle\langle -1|$ denotes the absorption of an excitation from time-dependent broadband squeezed vacuum field, and $\eta(t) = r \exp(i\phi)$ is time-dependent squeezing parameter with polar coordinates $\phi \in [0, 2\pi]$ and $r > 0$ called phase and amplitude of the squeezing, respectively. Form Eq.(24), the state

$$|\Phi_{\text{DF}}(r, \phi)\rangle = c(r)|-1\rangle - \exp(i\phi)s(r)|1\rangle, \quad (25)$$

with $c(r) = \cosh(r)/\sqrt{\cosh(2r)}$ and $s(r) = \sinh(r)/\sqrt{\cosh(2r)}$, satisfies $R(\eta)|\Phi_{\text{DF}}(t)\rangle = 0$. Here we assume that the phase of squeezing parameter is time-dependent with a linear trend $\phi = \omega_0 t$, and the amplitude of squeezing parameter is constant.

It has been shown in Refs.[19, 20] that the decoherence-free evolution can be achieved by adiabatically changing the squeezing phase. Comparing with Ref.[19], we will show here that the adiabatic limit can be removed when the time-dependent Hamiltonian in Eq.(23) is manipulated according to Eq.(22). In other words, the open system could evolve unitarily in the t-DFS by employing the scheme proposed in Sec.II. The master equation Eq.(23) can also be realized in the model of a pair of trapped four-level atoms[21], hence there are many manner to realize it in experiment.

In the following, the state $|\Phi_{\text{DF}}(t)\rangle$ is chosen as the basis of the t-DFS, whose complemental space is then spanned by $|\Phi_1^\perp(t)\rangle = s(r)|-1\rangle + \exp(i\omega_0 t)c(r)|1\rangle$ and

$|\Phi_2^\perp(t)\rangle = |0\rangle$. According to Eq.(22), the classical fields $\Omega_j(t)$ must be modulated as follow,

$$\begin{aligned} \Omega_1(t) &= \cosh(r) \exp(i\omega_0 t), \\ \Omega_2(t) &= \sinh(r), \\ \Omega_3(t) &= \omega_0 \sinh(r) \cosh(r) \exp(i\omega_0 t), \end{aligned} \quad (26)$$

such that $|\Phi_{\text{DF}}(t)\rangle$ forms a t-DFS.

The numerical results are presented in FIG.2, in which the initial state is chosen to be $|\Phi_{\text{DF}}(0)\rangle = c(r)|-1\rangle - s(r)|1\rangle$ with $r = 1$. We plot the purity $P(t) = \text{Tr}[\rho^2(t)]$ with the parameter $\omega_0 = 0.1\gamma$ (FIG.2(a)) and $\omega_0 = 10\gamma$ (FIG.2(b)) as a function of time. From FIG.2, we can see that the control fields $\Omega_j(t)$ play an important role in this scheme. Without the control fields (solid line in FIG.2), the decoherence-free evolution can be approximately realized only under the adiabatic limit, i.e., the smaller the ω_0 is, the better the state remains in the t-DFS. This can be understood by examining Eq.(22): When the control fields are absent, the t-DFS condition is equivalent to that the right side of Eq.(22) is zero, i.e., $\omega_0 \rightarrow 0$. In other words, to make sure that the evolution of quantum state is decoherence free, the squeezing phase has to change adiabatically. As expected, when the open system are controlled by the classical fields $\Omega_j(t)$ as Eq.(26) (dash line in FIG.2), the evolution of the quantum state is always unitary regardless of how fast the squeezing phase changes.

B. A toy model for DFS with time-dependent dimension

In this subsection, we present a toy model with varying dimension of t-DFS. Consider a five-level system coupled to two different broadband squeezing vacuum fields and six classical control fields (Ω_j and Ω'_j), as shown in FIG.3. $a(\omega)$ and $a'(\omega)$ are the annihilation operators of the environment with different polarization and frequency ω . Suppose the squeezing vacuum states for the modes $a(\omega)$ and $a'(\omega)$ have different squeezing parameters $\eta_1(t) = r_1 \exp(i\omega_0 t)$ and $\eta_2(t) = r_2 \exp(i\omega_0 t)$, respectively. Under the same assumptions used in Eq.(23), the master equation can be written as [19],

$$\begin{aligned} \dot{\rho}(t) &= -i[H(t), \rho(t)] \\ &+ \sum_{\alpha=1}^2 \gamma_\alpha \left[F_\alpha(t)\rho(t)F_\alpha^\dagger(t) - \frac{1}{2}\{F_\alpha^\dagger(t)F_\alpha(t), \rho(t)\} \right], \end{aligned} \quad (27)$$

where the Lindblad operators are $F_\alpha(t) = \cosh(r_\alpha)S_\alpha + \exp(i\omega_0 t)\sinh(r_\alpha)S_\alpha^\dagger$ ($\alpha = 1, 2$) with $S_1 = |1\rangle\langle 0| + |0\rangle\langle -1|$ and $S_2 = |1'\rangle\langle 0| + |0\rangle\langle -1'|$. The Hilbert space $\mathcal{H}_S(t)$ of open system can be spanned by the following orthogonal normalized bases,

$$|\Phi_{\text{DF}1}(t)\rangle = c_1|-1\rangle - \exp(i\omega_0 t)s_1|1\rangle, \quad (28)$$

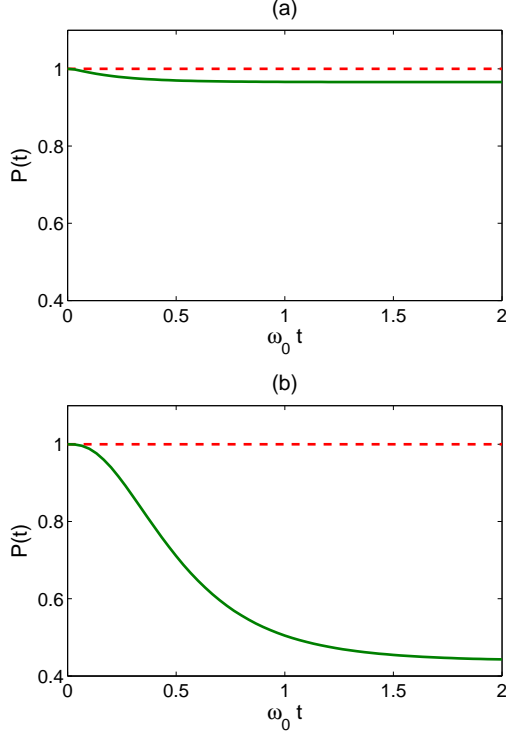


FIG. 2: (Color online) The purity $P(t)$ as a function of $\omega_0 t$ (in units of π) with time-dependent Hamiltonian (dash line) and time-independent Hamiltonian (solid line) for (a) $\omega_0 = 0.1\gamma$ and (b) $\omega_0 = 10\gamma$. The other parameters chosen are $\gamma = 1$, $r = 1$.

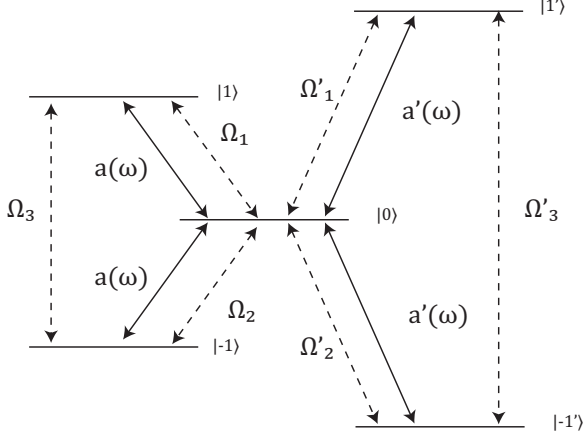


FIG. 3: (Color online) Schematic energy diagram. A five-level system, transition $|1\rangle \leftrightarrow |0\rangle$ and $|0\rangle \leftrightarrow |-1\rangle$ are driven by mode a , $|1'\rangle \leftrightarrow |0\rangle$ and $|0\rangle \leftrightarrow |-1'\rangle$ are driven by mode a' , and $|1'\rangle \leftrightarrow |-1'\rangle$ by b .

$$|\Phi_{\text{DF2}}(t)\rangle = c_2|-1'\rangle - \exp(i\omega_0 t)s_2|1'\rangle, \quad (29)$$

$$|\Phi_1^\perp(t)\rangle = s_1|-1\rangle + \exp(i\omega_0 t)c_1|1\rangle, \quad (30)$$

$$|\Phi_2^\perp(t)\rangle = s_2|-1'\rangle + \exp(i\omega_0 t)c_2|1'\rangle, \quad (31)$$

$$|\Phi_0^\perp(t)\rangle = |0\rangle, \quad (32)$$

with $s_i = \sinh(r_i)/\sqrt{\cosh(2r_i)}$ and $c_i = \cosh(r_i)/\sqrt{\cosh(2r_i)}$. It is not difficult to check that this model admits a two-dimensional t-DFS at most,

$$F_1(t)|\Phi_{\text{DF1}}\rangle = F_2(t)|\Phi_{\text{DF1}}\rangle = 0, \quad (33)$$

$$F_1(t)|\Phi_{\text{DF2}}\rangle = F_2(t)|\Phi_{\text{DF2}}\rangle = 0. \quad (34)$$

The Hamiltonian

$$H(t) = H_0(t) + T(t)\Omega(|\Phi_{\text{DF1}}(t)\rangle\langle\Phi_{\text{DF2}}(t)| + h.c.), \quad (35)$$

includes two terms: The first term $H_0(t)$ is required to construct the t-DFS with time-dependent dimension,

$$\begin{aligned} H_0(t) = & (\Omega_1(t)|1\rangle\langle 0| + \Omega_2(t)|0\rangle\langle -1| + \Omega_3(t)|1\rangle\langle -1| \\ & + \Omega_4(t)|1'\rangle\langle 0| + \Omega_5(t)|0\rangle\langle -1'| + \Omega_6(t)|1'\rangle\langle -1'| \\ & + h.c.), \end{aligned} \quad (36)$$

where the classical field strengths, according to Eq.(22), are designed as follow,

$$\begin{aligned} \Omega_1(t) &= \cosh(r_1)\exp(i\omega_0 t), \\ \Omega_2(t) &= \sinh(r_1), \\ \Omega_3(t) &= \omega_0 \sinh(r_1) \cosh(r_1) \exp(-i\omega_0 t), \\ \Omega'_1(t) &= \cosh(r_2)\exp(i\omega_0 t), \\ \Omega'_2(t) &= \begin{cases} 0, & \omega_0 t < \pi/2; \\ (1 + \cos(2\omega_0 t)) \sinh(r_2), & \pi/2 \leq \omega_0 t \leq \pi; \\ \sinh(r_2), & \omega_0 t > \pi. \end{cases} \\ \Omega'_3(t) &= \omega_0 \sinh(r_2) \cosh(r_2) \exp(-i\omega_0 t). \end{aligned} \quad (37)$$

This implies that, for $0 < \omega_0 t \leq \pi$, the t-DFS $\mathcal{H}_{\text{DFS}}(t)$ is one-dimensional and spanned by $|\Phi_{\text{DF1}}(t)\rangle$, since $\langle\Phi_i^\perp(t)|H_{\text{eff}}(t)|\Phi_{\text{DF1}}(t)\rangle = \langle\Phi_{\text{DF2}}(t)|H_{\text{eff}}(t)|\Phi_{\text{DF1}}(t)\rangle = 0$. For $\omega_0 t > \pi$, the DFS $\mathcal{H}_{\text{DFS}}(t)$ is two-dimensional spanned by $|\Phi_{\text{DF1}}(t)\rangle$, and $|\Phi_{\text{DF2}}(t)\rangle$, since $\langle\Phi_i^\perp(t)|H_{\text{eff}}(t)|\Phi_{\text{DF1}}(t)\rangle = \langle\Phi_i^\perp(t)|H_{\text{eff}}(t)|\Phi_{\text{DF2}}(t)\rangle = 0$. Thus, the dimension of the t-DFS $\mathcal{H}_{\text{DFS}}(t)$ changes with time. To illustrate the dimension changing with time, we introduce the second term of the Hamiltonian that induces a transition between $|\Phi_{\text{DF1}}(t)\rangle$ and $|\Phi_{\text{DF2}}(t)\rangle$, where $T(t) = \theta(\omega_0 t - \pi)$ is a step function and Ω is transition coefficient. This means that the transition is allowed for $\omega_0 t > \pi$ but it is forbidden for $0 < \omega_0 t \leq \pi$.

The numerical simulations are presented in FIG.4. The population on $|\Phi_{\text{DF1}}(t)\rangle$ (dash line) and $|\Phi_{\text{DF2}}(t)\rangle$ (solid line),

$$\begin{aligned} P_1(t) &= \langle\Phi_{\text{DF1}}(t)|\rho(t)|\Phi_{\text{DF1}}(t)\rangle, \\ P_2(t) &= \langle\Phi_{\text{DF2}}(t)|\rho(t)|\Phi_{\text{DF2}}(t)\rangle, \end{aligned} \quad (38)$$

are shown in FIG.4(a). As predicted, when $0 < \omega_0 t \leq \pi$, the population stays on $|\Phi_{\text{DF1}}(t)\rangle$; when $\omega_0 t > \pi$, the

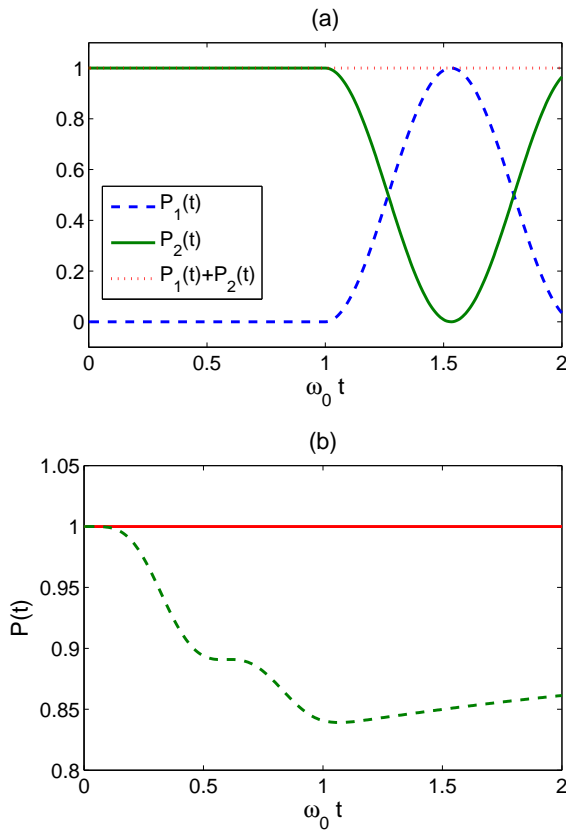


FIG. 4: (Color online) (a) The evolution of population on $|\Phi_{DF1}(t)\rangle$ (solid line), $|\Phi_{DF2}(t)\rangle$ (dash line) and population in $\mathcal{H}_{DFS}(t)$ (dots line) with $r_1 = r_2 = 1$ and $\Omega = \omega_0 = \gamma = 1$. (b) The evolution of the purity $P(t)$ with $T(t) = 1$ (dash line) and $T(t) = \theta(\omega_0 t - \pi)$ (solid line).

population transits between $|\Phi_{DF1}(t)\rangle$ and $|\Phi_{DF2}(t)\rangle$, but the total population $P_1(t) + P_2(t)$ (dot line in FIG.4(a)) remains unchanged, this means that the system does not leak out the t-DFS. If the Hamiltonian does not satisfy the t-DFS condition, for example, $T(t) = 1$ instead of $T(t) = \theta(\omega_0 t - \pi)$, the purity $P(t)$ will decay (dash line in FIG.4(b)). This implies that the dimension of the t-DFS really changes in the time evolution at time t_0 given by $\omega_0 t_0 = \pi$. Once the dimension of the t-DFS changes, the

time-independent DFS and t-DFS can not be connected by a unitary transformation. In the other words, the t-DFS is not trivial.

IV. CONCLUSION

In summary, we have defined and presented a necessary and sufficient condition for the time-dependent decoherence-free subspace (t-DFS) for open systems. In contrast to the time-independent DFS, the basis of this t-DFS is time-dependent. Besides, the dimension of t-DFS may change with time, this implies that we can not trivially get the t-DFS by unitary transformations. Two examples are presented to illustrate the t-DFS. In the first example, we show in details how to manipulate the Hamiltonian of a Ξ -type system to realize a one-dimensional t-DFS, while in the second example, we through a toy model to show that the dimension of the t-DFS can change with time. The later indicates that the t-DFS can not be derived by an unitary transformation from the conventional DFS.

The observation of the t-DFS and the prediction made for the t-DFS is within reach of recent technology. In fact, in a recent proposal [22], the authors proposed a scheme to entangle two atomic ensemble of Cesium at room temperature by engineering reservoir [23, 24]. These techniques together with measurement [23], can realize the t-DFS. For example, the time-dependent Lindblad operators may be achieved by modulating the detuning between the pumping field and the atom or by modifying the Zeeman splitting. (A detail relation between Lindblad operators and parameters of environment can be found in Refs. [22].) The parameters in the Lindblad operators can be modulated in experiment [25]. All these together can lead to the t-DFS. In practice, the Lindblad operators are not known *a priori*. However, we can re-design a decoherence-free subspace by controlling the Hamiltonian and engineering the reservoir. This means that we can improve the decoherence by designing a DFS to include the states of interests.

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